fixed coordinate system $Oxyz$. The particle's equations of motion are

$$x^{\cdot\cdot} + f_1 x = 0, \quad y^{\cdot\cdot} + f_2 y = 0, \quad z^{\cdot\cdot} + f_3 z = 0$$

The origin O is a trivial equilibrium position, unstable under conditions (3.10), (3.11). Let us assume that condition (3.9) is fulfilled. Then, according to what we have proved above, this equilibrium position can be made stable if to the potential forces acting on the particle we add a nonconservative force P perpendicular to the particle's radius-vector with projections on the coordinate axes $P_x = -p_1 y$, $P_y = p_1 x - p_3 z$, $P_z = p_3 y$, where p_1, p_3 satisfy system (3.14) ($p_2 = 0$).

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THE STRUCTURE OF SHOCK WAVES IN HYPERSONIC FLOWS

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The motion of gas in that region of curved hypersonic shock wave, where the angle of inclination τ of the latter to the velocity vector of the unperturbed stream is small, is analyzed with the use of Navier-Stokes equations. The number of terms retained in expansions of unknown functions in powers of τ is such as to permit the extension of solution into a new inviscid region by using the method of matching outer and inner asymptotic expansions. The statement of the problem in the new region is distinguished by that functions are specified at a point not by their values but by Taylor series.

1. Basic estimates and the form of asymptotic expansions in the region of the shock wave for $x \to \infty$. Let us consider the hypersonic flow of perfect gas of constant specific heats c_p and c_v. We denote the density of gas in the oncoming stream by ρ_∞ and its velocity directed along the x-axis of a Cartesian ($\nu = 1$ in the plane case) or cylindrical ($\nu = 2$ in the axisymmetric case) system of coordinates, by U_∞. We neglect the pressure of gas in the oncoming stream and set $P_\infty = 0$, which results in the Mach number $M_\infty = 0$. We define the dependence of coefficients of viscosity and thermal conductivity on enthalpy in the form of power functions $\lambda = \lambda_0 w^\omega$ and $k = k_0 w^\omega$, respectively. We denote the projections of the velocity vector on the x- and y-axes by v_x and v_y, respectively, the Prandtl number by N_{Pr}, and set $\varkappa = c_p / c_v$. It is convenient to specify subsequently the independent variables and the unknown functions in dimensionless terms, using ρ_∞, U_∞ and λ_0 as the basic dimensional units. As the basic system we take that of the Navier-Stokes equations in the dimensionless form

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + (\nu - 1)\frac{\rho v_y}{y} = 0, \quad p = \frac{\varkappa - 1}{\varkappa}\rho w \tag{1.1}$$

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[w^\omega\left(\frac{4}{3}\frac{\partial v_x}{\partial x} - \frac{2}{3}\frac{\partial v_y}{\partial y} - \frac{2}{3}(\nu - 1)\frac{v_y}{y}\right)\right] + \frac{\partial}{\partial y}\left[w^\omega\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + (\nu - 1)\frac{w^\omega}{y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[w^\omega\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{2}{3}\frac{\partial}{\partial y}\left[w^\omega\left(2\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x} - (\nu - 1)\frac{v_y}{y}\right)\right] + 2(\nu - 1)\frac{w^\omega}{y}\left(\frac{\partial v_y}{\partial y} - \frac{v_y}{y}\right)$$

$$\rho v_x \frac{\partial w}{\partial x} + \rho v_y \frac{\partial w}{\partial y} = v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\frac{w^\omega}{N_{Pr}}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{w^\omega}{N_{Pr}}\frac{\partial w}{\partial y}\right) + \frac{\nu - 1}{N_{Pr}}\frac{w^\omega}{y}\frac{\partial w}{\partial y} + 2w^\omega\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + (\nu - 1)\frac{v_y^2}{y^2}\right] + w^\omega\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 - \frac{2}{3}w^\omega\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + (\nu - 1)\frac{v_y}{y}\right]^2$$

It was shown in [1] that for flows at infinitely high Mach number the unperturbed region is separated from the perturbed one by the line of discontinuity of gasdynamic function derivatives. We shall call this line the shock wave front. We shall also use the estimates of gasdynamic functions within the shock wave structure, where the angle of inclination τ of the shock wave front to the oncoming stream velocity vector is small in comparison with unity. According to [1] we have

$$v_x = 1 + O(\tau^2) \qquad v_y = O(\tau), \qquad \rho = O(1) \tag{1.2}$$

$$p = O(\tau^2), \qquad w = O(\tau^2)$$

These estimates make it possible to determine the characteristic dimension N (measured along the normal to the shock wave front) of the region in which the flow is affected by viscosity and thermal conductivity, i.e. to estimate the thickness of the shock wave. Equating the order of the principal convection and viscosity terms in the second equation of the Navier-Stokes system (1.1), we obtain

$$N = O(\tau^{2\omega - 1}) \tag{1.3}$$

Let the equation of the shock wave front be of the form

$$y = bx^n, \qquad n < 1 \tag{1.4}$$

For $x \to \infty$ the angle of inclination of curve (1.4) tends to zero, i.e. estimates (1.2) and (1.3) are valid for considerable x. We substitute for y the new independent variable $\xi_1 = (1 - y / bx^n)x^\beta$, and select exponent β so that at distances of the order of $O(\tau^{2\omega-1})$ from line (1.4) the order of magnitude of ξ_1 becomes unity, i.e. ξ_1 becomes independent of x This immediately determines the exponent β, namely, $\beta = n - (n-1)(2w-1)$. Hence,

$$\xi_1 = \left(1 - \frac{y}{bx^n}\right) x^{n-(n-1)(2\omega-1)} \tag{1.5}$$

is to be taken as the variable which defines the flow within the shock wave structure.

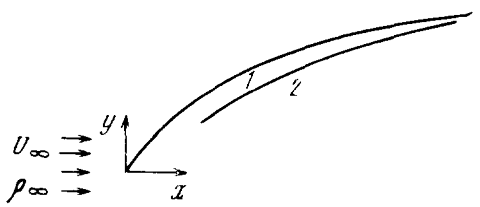

Fig. 1

In Fig. 1 this region is denoted by *1*. Note that for $\beta = 0$ the order of magnitude of ξ_1 becomes unity only in the case of $y \sim x^n$, when it is necessary to take into consideration the effects of viscosity and thermal conductivity throughout the whole region. If in preceding formulas $v_x = 1$ is assumed and t (time) is substituted for x, i.e. to pass to the nonstationary problem, then $\beta = 0$ corresponds to the exact self-similar solution of the Navier-Stokes equations with the variable ξ_1. The equation $\beta = 0$ determines the dependence of the exponent ω in the expression for the viscosity coefficient on n : $\omega = (1 - 2n) / (2 - 2n)$. This relationship is also obtained from the dimensional analysis of the independent constants of the problem [2]. The relative thickness N / y of the shock wave for $x \to \infty$ is proportional to $x^{-\beta}$, i.e. for $\beta > 0$ it tends to zero.

Let us consider the problem of extending the solution from the shock wave region 1 of characteristic dimension N and $\beta > 0$ into the new region and of formulating the problem in that region. In subsequent computations, except in the Appendix, we assume

$$n = 2 / (2 + \nu), \qquad \omega = 1$$

Although in the considered problem these values of n and ω are not specific to it, but from the physical point of view such choice of n corresponds to the problem of hypersonic flow past a finite body (in a nonstationary formulation it corresponds to that of strong explosion). The choice of ω conforms to that in [3], where the motion in the hypersonic trail downstream of the body is analyzed in a similar formulation.

Let us specify along surface (1.4) the following conditions:

$$\rho = v_x = 1, \qquad v_y = p = w = 0 \tag{1.6}$$

$$\lambda \frac{\partial v_x}{\partial N} = \lambda \frac{\partial v_y}{\partial N} = k \frac{\partial w}{\partial N} = 0$$

We seek the solution in the region of the shock wave in the form of series expansions in powers of x with coefficients which are functions of the self-similar variable ξ_1. Since expansions of the unknown functions for plane and axisymmetric flows are different, we shall consider them separately. Let us first consider plane-parallel flows. In accordance with estimates (1.2) we have

$$
\begin{aligned}
v_x &= 1 - \frac{8b^2}{9(\varkappa+1)} x^{-2/3} [U_{11}(\xi_1) + x^{-2/3} U_{12}(\xi_1) + x^{-1} U_{13}(\xi_1)] \qquad (1.7)\\
v_y &= \frac{4}{3} \frac{b}{\varkappa+1} x^{-1/3} [V_{11}(\xi_1) + x^{-2/3} V_{12}(\xi_1) + x^{-1} V_{13}(\xi_1)]\\
\rho &= \frac{\varkappa+1}{\varkappa-1} [R_{11}(\xi_1) + x^{-2/3} R_{12}(\xi_1) + x^{-1} R_{13}(\xi_1)]\\
p &= \frac{8}{9} \frac{b^2}{\varkappa+1} x^{-2/3} [P_{11}(\xi_1) + x^{-2/3} P_{12}(\xi_1) + x^{-1} P_{13}(\xi_1)]\\
w &= \frac{8}{9} \frac{b^2\varkappa}{(\varkappa+1)^2} x^{-2/3} [W_{11}(\xi_1) + x^{-2/3} W_{12}(\xi_1) + x^{-1} W_{13}(\xi_1)]
\end{aligned}
$$

The order of principal terms determined by the powers of x in expansions (1.7) are selected in accordance with estimates (1.2), and the order of the second and third terms has been so chosen as to compensate in the Navier-Stokes equations the products of functions with subscripts 11 which do not appear in the equations of the first approximation. Conditions (1.6) may be restated as conditions imposed on functions of various approximations at point $\xi_1 = 0$

$$
\begin{aligned}
&U_{11} = V_{11} = P_{11} = W_{11} = 0, \qquad R_{11} = (\varkappa - 1)/(\varkappa + 1) \qquad (1.8)\\
&U_{12} = V_{12} = R_{12} = P_{12} = W_{12} = 0\\
&U_{13} = V_{13} = R_{13} = P_{13} = W_{13} = 0\\
&W_{11} \frac{dU_{11}}{d\xi_1} = W_{11} \frac{dV_{11}}{d\xi_1} = W_{11} \frac{dW_{11}}{d\xi_1} = 0\\
&W_{11} \frac{dU_{12}}{d\xi_1} + W_{12} \frac{dU_{11}}{d\xi_1} = W_{11} \frac{dV_{12}}{d\xi_1} + W_{12} \frac{dV_{11}}{d\xi_1} = W_{11} \frac{dW_{12}}{d\xi_1} + W_{12} \frac{dW_{11}}{d\xi_1} = 0\\
&W_{11} \frac{dU_{13}}{d\xi_1} + W_{13} \frac{dU_{11}}{d\xi_1} = W_{11} \frac{dV_{13}}{d\xi_1} + W_{13} \frac{dV_{11}}{d\xi_1} = W_{11} \frac{dW_{13}}{d\xi_1} + W_{13} \frac{dW_{11}}{d\xi_1} = 0
\end{aligned}
$$

In addition to conditions (1.8) we specify that for $\xi_1 \to \infty$ the first approximation functions must tend to constant values and that none of the second and third approximation functions may exponentially increase.

2. Systems of equations of various approximations and the asymptotics of their solutions for $\xi_1 \to \infty$. Substituting expansions (1.7) into the system of Eqs. (1.1) and integrating once each of these with allowance for (1.8), we obtain

$$
\begin{aligned}
&\frac{\varkappa+1}{2} R_{11} - R_{11} V_{11} = \frac{\varkappa-1}{2}, \qquad P_{11} = R_{11} W_{11} \qquad (2.1)\\
&U_{11} - P_{11} - \frac{4}{3} \frac{\varkappa}{(\varkappa+1)^2} W_{11} \frac{dU_{11}}{d\xi_1} - \frac{4}{9} \frac{\varkappa}{(\varkappa+1)^2} W_{11} \frac{dV_{11}}{d\xi_1} = 0\\
&V_{11} - P_{11} - \frac{16}{9} \frac{\varkappa}{(\varkappa+1)^2} W_{11} \frac{dV_{11}}{d\xi_1} = 0\\
&V_{11}^2 - W_{11} + \frac{4}{3} \frac{\varkappa^2}{(\varkappa+1)^2} \frac{1}{N_{Pr}} W_{11} \frac{dW_{11}}{d\xi_1} = 0
\end{aligned}
$$

System (2.1) consists of two finite relationships and three differential equations with the equation for the longitudinal velocity U_{11} separated and solved after all remaining functions have been determined. Eliminating R_{11} and P_{11}, we obtain a system of two first order differential equations for W_{11} and V_{11}. This system is equivalent to the system arising in the analysis of the structure of a one-dimensional shock wave in a viscous and heat conducting gas, which was considered by numerous authors and was first analyzed in [4]. For $N_{\mathrm{Pr}} = 3/4$ Eqs. (2.1) admit an analytic solution [4] which satisfies conditions (1.8) and for $\xi_1 \to \infty$ also conditions

$$U_{11} = V_{11}, \quad R_{11} = (\varkappa - 1) / (\varkappa + 1 - 2V_{11})$$

$$W_{11} = V_{11}(\varkappa + 1 - V_{11}) / \varkappa$$

$$P_{11} = \frac{\varkappa - 1}{\varkappa} \frac{V_{11}(\varkappa + 1 - V_{11})}{\varkappa + 1 - 2V_{11}}, \quad \xi_1 = \frac{16\varkappa}{9(\varkappa + 1)^2}[(3\varkappa + 1)V_{11} - V_{11}^2 - \varkappa(\varkappa - 1)\ln(1 - V_{11})]$$

For other N_{Pr} the solution has to be derived numerically, although the relationship $U_{11} = V_{11}$ remains valid, which can be readily ascertained by the analysis of the second and third of Eqs. (2.1). For any N_{Pr} the asymptotics of the first approximation functions for $\xi_1 \to \infty$ are of the form

$$U_{11} = 1 + \mathrm{TST}, \quad V_{11} = 1 + \mathrm{TST}, \quad R_1 = 1 + \mathrm{TST} \tag{2.2}$$

$$P_{11} = 1 + \mathrm{TST}, \quad W_{11} = 1 + \mathrm{TST}$$

where TST denotes, as usual, exponentially attenuated terms.

Taking into consideration conditions (1.8), we write the system of second approximation equations as

$$\frac{\varkappa + 1}{2} R_{12} - R_{11}V_{12} - R_{12}V_{11} = \frac{4}{9} b^2 R_{11} U_{11} \tag{2.3}$$

$$U_{12} - P_{12} - \frac{4}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dV_{12}}{d\xi_1} - \frac{4}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{12}\frac{dV_{11}}{d\xi_1} - \frac{4}{3}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dU_{12}}{d\xi_1} - \frac{4}{3}\frac{\varkappa}{(\varkappa+1)^2} W_{12}\frac{dU_{11}}{d\xi_1} = \frac{64}{81}\frac{\varkappa b^2}{(\varkappa+1)^2} W_{11}\frac{dU_{11}}{d\xi_1}$$

$$V_{12} - P_{12} - \frac{16}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dV_{12}}{d\xi_1} - \frac{16}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{12}\frac{dV_{11}}{d\xi_1} = \frac{16}{27}\frac{\varkappa b^2}{(\varkappa+1)^2} W_{11}\frac{dV_{11}}{d\xi_1} + \frac{16}{81}\frac{\varkappa b^2}{(\varkappa+1)^2} W_{11}\frac{dU_{11}}{d\xi_1}$$

$$2V_{11}V_{12} - W_{12} + \frac{4}{3}\frac{\varkappa^2}{(\varkappa+1)^2}\frac{1}{N_{Pr}}\left(W_{11}\frac{dW_{12}}{d\xi_1} + W_{12}\frac{dW_{11}}{d\xi_1}\right) = -\frac{4}{9} b^2 U_{11}^2 - \frac{16}{27}\frac{\varkappa^2 b^2}{(\varkappa+1)^2} W_{11}\frac{dW_{11}}{d\xi_1}$$

$$P_{12} = W_{11}R_{12} + W_{12}R_{11}$$

As in the first approximation system, the equation which defines U_{12} in system (2.3) is separated. Eliminating R_{12} and P_{12} with the use of finite relationships, for W_{12} and V_{12} we obtain the system of two linear differential equations

$$-\frac{16}{9}\frac{\varkappa}{(\varkappa+1)^2}W_{11}\frac{dV_{12}}{d\xi_1}+V_{12}\left(1-\frac{2P_{11}}{\varkappa+1-2V_{11}}\right)- \tag{2.4}$$

$$W_{12}\left(\frac{16}{9}\frac{\varkappa}{(\varkappa+1)^2}\frac{dV_{11}}{d\xi_1}+R_{11}\right)=\frac{16}{27}\frac{b^2\varkappa}{(\varkappa+1)^2}W_{11}\frac{dV_{11}}{d\xi_1}+$$

$$\frac{16}{81}\frac{b^2\varkappa}{(\varkappa+1)^2}W_{11}\frac{dU_{11}}{d\xi_1}+\frac{8}{9}\frac{b^2U_{11}P_{11}}{\varkappa+1-2V_{11}}$$

$$\frac{4}{3}\frac{\varkappa}{(\varkappa+1)^2}\frac{1}{N_{Pr}}W_{11}\frac{dW_{12}}{d\xi_1}+\frac{2}{\varkappa}V_{11}V_{12}+W_{12}\left(-\frac{1}{\varkappa}+\right.$$

$$\left.\frac{4}{3}\frac{\varkappa}{(\varkappa+1)^2}\frac{1}{N_{Pr}}\frac{dW_{11}}{d\xi_1}\right)=-\frac{4}{9}\frac{b^2}{\varkappa}U_{11}{}^2-\frac{16}{27}\frac{b^2\varkappa}{(\varkappa+1)^2}\frac{1}{N_{Pr}}W_{11}\frac{dW_{11}}{d\xi_1}$$

System (2.4) has the particular solution

$$V_{21}=-{}^4/_9b^2V_{11},\qquad W_{12}=-{}^4/_9b^2W_{11} \tag{2.5}$$

which satisfies the conditions for $\xi_1=0$ and $\xi_1\to\infty$. Point $\xi_1=0$ is singular for the system (2.4), and formulas (1.8) do not unambigously define the solution. Let us show that the supplementary requirement for the absence of exponential increase for $\xi_1\to\infty$ makes it possible to isolate an unambigous solution. Let us examine the homogeneous system obtained from (2.4) by rejecting its right-hand parts which depend on first approximation functions. Simple reasoning shows that the first integral of the homogeneous system has

$$V_{12}'=dV_{11}/d\xi_1,\qquad W_{12}'=dW_{11}/d\xi_1 \tag{2.6}$$

Using solution (2.6), we can reduce system (2.4) to a first order linear differential equation for functions V_{12}'' and obtain the second linearly independent solution

$$V_{12}''=\frac{dV_{11}}{d\xi_1}\int\limits_{0}^{\xi_1}V_{11}W_{11}^{-2+3\varkappa/4N_{Pr}}\left(\frac{dV_{11}}{d\xi_1}\right)^{-2}\exp\left\{\int\limits_0^{\xi_1}\frac{9}{16}\frac{(\varkappa+1)^2}{\varkappa}\times\right. \tag{2.7}$$

$$\left.\left[\left(1+\frac{4}{3}\frac{N_{Pr}}{\varkappa}\right)\frac{V_{11}{}^2}{W_{11}{}^2}-\frac{2}{\varkappa-1}R_{11}{}^2\right]d\xi_2\right\}d\xi_1$$

The complete solution for function V_{12} is of the form

$$V_{12}=-{}^4/_9b^2V_{11}+C_1V_{12}'+C_2V_{12}'$$

The solutions for functions R_{12}, P_{12} and W_{12} are of a similar form. Let us apply the asymptotics of first approximation functions derived in the Appendix, to the analysis of asymptotic properties of V_{12}' and V_{12}''. Computations show that for $\xi_1\to 0$

$$V_{12}'=\begin{cases}\text{const}\neq 0 & \text{for } 3\varkappa/4N_{Pr}>1\\ \infty & \text{for } 3\varkappa/4N_{Pr}<1\end{cases},\qquad V_{12}''\to 0$$

For $\xi_1\to\infty$ function V_{12}' exponentially decreases, while V_{12}'' exponentially increases. Hence the conditions for $\xi_1=0$ and $\xi_1\to\infty$ make it necessary to set $C_1=0$ and $C_2=0$. The asymptotic properties of U_{12} are analyzed in the same way. As the result, we obtain the solution of system (2.3) which for $\xi_1=0$ satisfies conditions (1.8) and for $\xi_1\to\infty$ the conditions for the absence of exponential increase

$$U_{12}=-{}^4/_9b^2U_{11},\quad V_{12}=-{}^4/_9b^2V_{11},\quad R_{12}=0 \tag{2.8}$$

$$P_{12}=-{}^4/_9b^2P_{11},\quad W_{12}=-{}^4/_9b^2W_{11}$$

Formulas (2.3) and (2.8) make it possible to determine the asymptotic behavior of second approximation functions for $\xi_1 \to \infty$, namely

$$U_{12} = -{}^4/_9 b^2 + \text{TST}, \quad V_{12} = -{}^4/_9 b^2 + \text{TST}, \quad R_{12} = 0 \tag{2.9}$$
$$P_{12} = -{}^4/_9 b^2 + \text{TST}, \quad W_{12} = -{}^4/_9 b^2 + \text{TST}$$

Taking into account conditions (1.8), for the third approximation functions we obtain

$$\frac{\varkappa+1}{2} R_{13} - R_{11}V_{13} - R_{13}V_{11} = -\frac{\varkappa+1}{4}\int_0^{\xi_1} \zeta \frac{dR_{11}}{d\zeta}\, d\zeta \tag{2.10}$$

$$U_{13} - P_{13} - \frac{4}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dV_{13}}{d\xi_1} - \frac{4}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{13}\frac{dV_{11}}{d\xi_1} -$$
$$\frac{4}{3}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dU_{13}}{d\xi_1} - \frac{4}{3}\frac{\varkappa}{(\varkappa+1)^2} W_{13}\frac{dU_{11}}{d\xi_1} =$$
$$-\frac{\varkappa+1}{2(\varkappa-1)} U_{11}\int_0^{\xi_1} R_{11}\, d\zeta + \int_0^{\xi_1}\Big[\frac{3}{2}\frac{\varkappa+1}{\varkappa-1} R_{11}U_{11} - P_{11} +$$
$$\frac{1}{2}\zeta\frac{dP_{11}}{d\zeta} + \frac{8}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dV_{11}}{d\zeta} - \frac{4}{9}\frac{\varkappa}{(\varkappa+1)^2} V_{11}\frac{dW_{11}}{d\zeta}\Big]\, d\zeta$$

$$V_{13} - P_{13} - \frac{16}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{11}\frac{dV_{13}}{d\xi_1} - \frac{16}{9}\frac{\varkappa}{(\varkappa+1)^2} W_{13}\frac{dV_{11}}{d\xi_1} =$$
$$\frac{\varkappa+1}{2(\varkappa-1)}\Big(2\int_0^{\xi_1} R_{11}V_{11}\, d\zeta + V_{11}\int_0^{\xi_1} R_{11}\, d\zeta\Big)$$

$$2V_{11}V_{13} - W_{13} + \frac{4}{3}\frac{1}{N_{Pr}}\frac{\varkappa^2}{(\varkappa+1)^2}\Big(W_{11}\frac{dW_{13}}{d\xi_1} + W_{13}\frac{dW_{11}}{d\xi_1}\Big) =$$
$$\frac{\varkappa+1}{2(\varkappa-1)}\Big[W_{11}\int_0^{\xi_1} R_{11}\, d\zeta - 3\int_0^{\xi_1}(R_{11}W_{11} + \varkappa R_{11}V_{11}^2)\, d\zeta +$$
$$4\varkappa V_{11}\int_0^{\xi_1} R_{11}V_{11}\, d\zeta - \varkappa V_{11}^2\int_0^{\xi_1} R_{11}\, d\zeta, \quad P_{13} = W_{11}R_{13} + W_{13}R_{11}$$

A feature of Eqs. (2.10) is that they differ from second approximation equations (2.3) only by their right-hand parts, i.e. the solutions of the homogeneous systems are the same. We take functions (2.6) and (2.7) for these. Using solutions of the homogeneous system (2.10), it is not difficult to derive the solution of the nonhomogeneous system, which satisfies the conditions for $\xi_1 = 0$ and $\xi_1 \to \infty$. The asymptotics of that solution are of the form (2.11)

$$U_{13} = -\frac{2\varkappa-3}{\varkappa-1}\xi_1 + O(1), \quad V_{13} = -\frac{3}{2}\xi_1 + O(1), \quad R_{13} = -\frac{3}{\varkappa-1}\xi_1 + O(1)$$
$$P_{13} = -\frac{2\varkappa-1}{\varkappa-1}\xi_1 + O(1), \quad W_{13} = \frac{2(2-\varkappa)}{\varkappa-1}\xi_1 + O(1)$$

3. Transition to inner region and statement of problem in that region. Let us compare now the principal terms of expansions of function with subscript 13 defined by equalities (2.11) with the principal terms of expansions of functions with subscripts 11 defined in (2.3) (such comparison is possible in this problem, since second approximation functions do not appear in the system of third approximation functions). It will be seen that for $\xi_1 \to \infty$ the product $x^{-1}\xi_1$ can be of the order of unity, even when $x \to \infty$. This condition defines the new self-similar variable

$$\xi_2 = y \,/\, (bx^{2/3}) \tag{3.1}$$

and the new region denoted by 2 in Fig. 1 in which ξ_2 is of the order of unity. We use the method of matching outer and inner asymptotic expansions for constructing the solution in region 2. Using the definition (3.1) of the self-similar variable in this region, we seek a solution of the form

$$\begin{aligned}
v_x &= 1 - \frac{8b^2}{9(\varkappa+1)} x^{-2/3} [U_{21}(\xi_2) + x^{-2/3} U_{22}(\xi_2)] \\
v_y &= \frac{4}{3}\frac{b}{\varkappa+1} x^{-1/3} [V_{21}(\xi_2) + x^{-2/3} V_{22}(\xi_2)] \\
\rho &= \frac{\varkappa+1}{\varkappa-1} [R_{21}(\xi_2) + x^{-2/3} R_{22}(\xi_2)] \\
p &= \frac{8}{9}\frac{b^2}{\varkappa-1} x^{-2/3} [P_{21}(\xi_2) + x^{-2/3} P_{22}(\xi_2)] \\
w &= \frac{8}{9}\frac{b^2\varkappa}{(\varkappa+1)^2} x^{-2/3} [W_{21}(\xi_2) + x^{-2/3} W_{22}(\xi_1)]
\end{aligned} \tag{3.2}$$

Formulas (2.2), (2.9) and (2.11) make it possible to determine the form of unknown functions for $\xi_2 \to 1$

$$\begin{aligned}
&U_{21} = 1 + \frac{2(\varkappa-3)}{\varkappa-1}(\xi_2 - 1) + \cdots, \qquad V_{21} = 1 + \frac{3}{2}(\xi_2 - 1) + \cdots \\
&R_{21} = 1 + \frac{3}{\varkappa-1}(\xi_2 - 1) + \cdots, \qquad P_{21} = 1 + \frac{2\varkappa-1}{\varkappa-1}(\xi_2 - 1) + \cdots \\
&W_{21} = 1 - \frac{2(2-\varkappa)}{\varkappa-1}(\xi_2 - 1) + \cdots \\
&U_{22} = 1 + \cdots, \quad V_{22} = 1 + \cdots, \quad R_{22} = 0 + \cdots, \quad P_{22} = 1 + \cdots, \\
&W_{22} = 1 + \cdots
\end{aligned} \tag{3.3}$$

Let us consider in detail the first approximation functions. Substituting expansions (3.2) into the system of Eqs. (1.1) and retaining the principal terms with respect to x, we obtain the known system of differential equations

$$\begin{aligned}
&\left(V_{21} - \frac{\varkappa+1}{2}\xi_2\right)\frac{dR_{21}}{d\xi_2} + R_{21}\frac{dV_{21}}{d\xi_2} = 0 \\
&\left(V_{21} - \frac{\varkappa+1}{2}\xi_2\right)R_{21}\frac{dV_{21}}{d\xi_2} + \frac{\varkappa-1}{2}\frac{dP_{21}}{d\xi_{21}} - \frac{\varkappa+1}{4}R_{21}V_{21} = 0 \\
&\left(V_{21} - \frac{x+1}{2}\xi_2\right)\frac{dP_{21}}{d\xi_2} + \varkappa P_{21}\frac{dV_{21}}{d\xi_2} - \frac{\varkappa+1}{4}P_{21} = 0 \\
&P_{21} = R_{21}W_{21}, \qquad U_{21} = \frac{1}{\varkappa+1}\left(V_{21}^2 + \varkappa\frac{P_{21}}{R_{21}}\right)
\end{aligned} \tag{3.4}$$

which define a strong explosion [5 – 7]. However (3.4) is a system of first order differential equations for which it is sufficient to know the values of functions at point $\xi_1 = 1$, since the latter determine the solution completely. The second terms in expansion (3.3) represent additional conditions which are actually imposed on the derivatives. The solution of the problem of strong explosion can be expanded at point $\xi_1 = 1$ into the Taylor series

$$U_{21} = 1 + \frac{dU_{21}}{d\xi_2}(\xi_2 - 1) + \cdots, \quad \cdots, W_{21} = 1 + \frac{dW_{21}}{d\xi_2}(\xi_2 - 1) + \cdots \quad (3.5)$$

There remains to compare expansions (3.3) and (3.5). Such comparison shows that the derivatives $(dU_{21} / d\xi_2)_{\xi_2=1}, \ldots, (dW_{21} / d\xi_2)_{\xi_2=1}$, in fact conform to specifications set by formulas (3.3). This is not a coincidence, and is explained by the analytic properties of solutions of the Navier-Stokes equations which are of the elliptic kind. It is obvious that by continuing the computation of subsequent terms of expansions of unknown functions (1.7) in decreasing powers of x in the region of shock wave 1 it is possible to expand into Taylor series functions with subscripts 21, 22, ..., which are definable in (3.2) in the neighborhood of point $\xi_2 = 1$ of region 2.

A linear system of differential equations is obtained for second approximation functions in region 2, and conditions (3.3) make it possible to formulate the Cauchy problem for that system. In this case three terms of expansion (1.7) made it possible to calculate for functions with subscripts 22 only the values of these at point $\xi_2 = 1$.

In concluding the analysis of the plane case we note that the first two terms of expansions (3.2) in region 2 for $\omega = 1$ and $n = {}^2/_3$ can be derived by substituting the gas-dynamic shock wave (1.4) with the Hugoniot condition along it for the discontinuity line of derivatives (1.4) and the whole region 1.

The basic aim of this work is not simply the calculation of corrections to the solution in the region of a curved shock wave [8, 9] but to show how the solution in the new region, where the effect of viscosity and thermal conductivity can be neglected in the principal terms, is formed by the solution in the region of the shock wave.

4. Axisymmetric flow. In the case of axial symmetry for $\omega = 1$ we have $\nu = 2$, $n = {}^1/_2$, and the variable ξ_1 in accordance with (1.5) is of the form $\xi_1 = [1 - y / (bx^{1/2})]x$. We write the expansions of functions in region 1 as

$$v_x = 1 - \frac{b^2}{2(\varkappa + 1)} x^{-1} [U_{11}(\xi_1) + x^{-1} U_{12}(\xi_1)] \quad (4.1)$$

$$v_y = \frac{b}{\varkappa + 1} x^{-1/2} [V_{11}(\xi_1) + x^{-1} V_{12}(\xi_1)]$$

$$\rho = \frac{\varkappa + 1}{\varkappa - 1} [R_{11}(\xi_1) + x^{-1} R_{12}(\xi_1)]$$

$$p = \frac{b^2}{2(\varkappa + 1)} x^{-1} [P_{11}(\xi_1) + x^{-1} P_{12}(\xi_1)]$$

$$w = \frac{b^2 \varkappa}{2(\varkappa + 1)^2} x^{-1} [W_{11}(\xi_1) + x^{-1} W_{11}(\xi_1)]$$

As in the analysis of plane-parallel flows, the order of principal terms is chosen here in conformity with estimates (1.2), while the order of second terms is such as to compensate discrepancies generated in the Navier-Stokes equations by the principal terms. However, unlike in expansions (1.7), only two terms are retained in expansions (1.4). We stipulate the fulfillment of conditions (1.8) at point $\xi_1 = 0$, and that for $\xi_1 \to \infty$ the first approximation functions must tend to constant values and that none of the second approximation functions can exponentially increase. These requirements uniquely define solutions $U_{11}, \ldots, W_{12}$. Analysis of the asymptotics of these solutions for

$\xi_1 \to \infty$ shows the necessity of introducing the new region 2 whose characteristic variable is

$$\xi_2 = y \,/\, (bx^{1/2}) \tag{4.2}$$

Unlike in the case of plane-parallel flow for the introduction of the new variable in the case of axisymmetric flow it is sufficient to consider only two terms in the expansion (4.1). For the principal terms of expansion in region 2, where the variable ξ_2 defined by equality (4.2) is of the order of unity, we obtain not only the functions themselves, but also their first derivatives in the form of boundary conditions for $\xi_2 \to 1$.

When formulating the problem in region 2 only for the principal terms it is possible to specify Hugoniot's conditions along line (1.4). For second approximation functions these conditions differ from the latter and are derived from the analysis of asymptotics of functions with subscripts 12 for $\xi_1 \to \infty$.

Appendix. For any arbitrary ω and n the system of equations for first approximation functions in region 1 can be reduced to the form

$$\frac{\varkappa - 1}{\varkappa + 1 - 2V_{11}} W_{11} - V_{11} + \frac{4}{3}\frac{1}{b^2 n}\left[2b^2 n^2 \frac{\varkappa}{(\varkappa + 1)^2}\right]^{\omega} W_{11}{}^{\omega} \frac{dV_{11}}{d\xi_1} = 0 \tag{A.1}$$

$$V_{11}{}^2 - W_{11} - \frac{\varkappa}{N_{Pr} b^2 n}\left[2b^2 n^2 \frac{\varkappa}{(\varkappa + 1)^2}\right]^{\omega} W_{11}{}^{\omega} \frac{dW_{11}}{d\xi_1} = 0$$

with initial data $W_{11} = V_{11} = 0$, when $\xi_1 = 0$. However this point is not singular; it can be readily shown by passing to the phase variable V_{11} that it is a saddle point. The behavior of functions in its vicinity depends on parameter $3\varkappa / 4 N_{Pr}$. For $3\varkappa / 4 N_{Pr} >$ 1 we have

$$V_{11} = \frac{\varkappa - 1}{\varkappa + 1}\frac{A_w}{1 - 4N_{Pr} / 3\varkappa}\xi_1^{1/\omega} + a_V \xi_1^{3\varkappa / (4\omega N_{Pr})} + O(\xi_1^{2/\omega}) \tag{A.2}$$

$$W_{11} = A_w \xi_1^{1/\omega} + O(\xi_1^{2/\omega}), \qquad A_w = \frac{(\varkappa + 1)^2}{2b^2 n^2 \varkappa}\left(\frac{b^2 n N_{Pr}}{\varkappa}\right)^{1/\omega}$$

where a_V is an arbitrary constant.

For $1/2 < 3\varkappa / 4 N_{Pr} < 1$ the first and second terms in formulas (A.2) for V_{11} change their places, while in the expansion for W_{11} the form of the principal term remains unchanged. The estimate of terms omitted in the expressions for V_{11} and W_{11} is altered and becomes $O(\xi_1^{1/\omega + 3\varkappa / (4\omega N_{Pr})})$. for both functions.

For $3\varkappa / 4 N_{Pr} = 1$ the principal term of expansion V_{11} is proportional to $\xi_1{}^{1/\omega} \ln \xi_1$, and for $3\varkappa / 4 N_{Pr} = 1/2$ it is proportional to $\xi_1{}^{1/2\omega} \ln \xi_1$.

For $3\varkappa / 4 N_{Pr} < 1/2$ we have

$$V_{11} = [(1/2 - 3\varkappa / 4N_{Pr})\, 2A_w{}']^{1/2} \xi_1^{1/2\omega} + a_V{}' \xi_1^{(4N_{Pr} - 3\varkappa)/(6\omega\varkappa)} + O(\xi_1^{3/2\omega}) \tag{A.3}$$

$$W_{11} = A_w{}' \xi_1^{1/\omega} + a_w{}' \xi_1^{2N_{Pr}/(3\omega\varkappa)} + O(\xi_1^{2/\omega})$$

$$A_w{}' = \frac{1}{2b^2 n^2}\frac{(\varkappa + 1)^2}{\varkappa}\left(\frac{3b^2 n\omega}{2}\right)^{1/\omega}, \qquad a_w{}' = -\frac{4N_{Pr}}{3\omega\varkappa}\left[\left(\frac{1}{2} - \frac{3\varkappa}{4N_{Pr}}\right) 2A_w{}'\right]^{1/2} a_V{}'$$

where a_V' is an arbitrary constant.

Some of these expansions were known earlier. Thus asymptotics of the kind of (A.3) were indicated in [10] in the solution of the self-similar problem of a uniformly accelerated plate with $\omega = 3/2$. Later, these asymptotics were used in the range $1/2 < 3\varkappa / 4 N_{Pr} < 1$ which is determined by the requirement for the term with the arbitrary coefficient to be the principal one in the expansion of at least one of the functions. As shown by expansions (A.2) and (A.3) this is not so for $3\varkappa / 4\ N_{Pr}$ outside that range.

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ON A CLASS OF SOLUTIONS OF THE NONLINEAR EQUATION FOR THE VELOCITY POTENTIAL

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We construct a class of exact solutions of the equation for the velocity potential of unsteady plane flows of a polytropic gas. These solutions contain an infinite number of arbitrary functions of a single argument. They are given in the form of series in rational powers of the characteristic argument in the space of variables of the time-velocity hodograph. We study the applications of the series obtained to solving certain problems of flows arising during the motion of curvilinear pistons through a gas, so that at the initial instant the normal velocity and